

LEHMER PAIRS AND DERIVATIVES OF HARDY'S Z-FUNCTION

ALEKSANDER SIMONIĆ

ABSTRACT. Occurrences of very close zeros of the Riemann zeta function are in strong connection with Lehmer pairs and with the Riemann Hypothesis. The object of the present paper is to derive the condition on a pair of consecutive simple zeros of the ζ -function to be a Lehmer pair in terms of derivatives of Hardy's Z -function. Furthermore, we demonstrate on a concrete example that this provides a new method of verification that a given pair of zeros is Lehmer by means of numerical integration in a neighborhood of those zeros.

1. INTRODUCTION AND MAIN RESULT

The Riemann zeta function $\zeta(s) := \sum_{n=1}^{\infty} n^{-s}$ for $\Re\{s\} > 1$ is one of the most studied objects in all mathematics. Introduced by Euler as a real-valued function, Riemann first observed intimate connection between zeros of the complex-valued zeta function and prime numbers. In the famous 1859 paper he analytically continued $\zeta(s)$ to $\mathbb{C} \setminus \{1\}$ and wrote a conjecture which later becomes known as the *Riemann Hypothesis*: *all zeros of the ζ -function in the strip $0 \leq \Re\{s\} \leq 1$ have real part $1/2$; this line is called the *critical line*. Standard references on the subject are [Edw01, Tit86].*

A very useful function in investigation of the Riemann Hypothesis is *Hardy's Z -function* $Z(t)$, which is for real t defined by

$$Z(t) := e^{i\vartheta(t)} \zeta\left(\frac{1}{2} + it\right), \quad \vartheta(t) := \frac{1}{2i} \log \frac{\Gamma\left(\frac{1}{4} + i\frac{t}{2}\right)}{\Gamma\left(\frac{1}{4} - i\frac{t}{2}\right)} - \frac{\log \pi}{2} t.$$

This function is also known as the *Riemann-Siegel Z -function* or simply the *Z -function*. Hardy's Z -function is a smooth real function and its zeros are precisely ordinates of zeros of the ζ -function along the critical line. Furthermore, $Z(t)$ is an even function, so it is enough to study its behavior on the interval $[0, \infty)$. For more information consult Ivić's monograph [Ivi13].

Huge step in computation of zeros was made in 1956 by D. H. Lehmer who computed the first 25000 zeros. He observed that some zeros are extremely close relative to others. The first such pair consists of $\rho_{6709} = 0.5 + 7005.0629i$ and $\rho_{6710} = 0.5 + 7005.1006i$. Average spacing between consecutive zeros up to 25000th zero is about 0.88, but distance of these two zeros is not greater than 0.04. Occurrence of such pairs of close zeros is known as *Lehmer's phenomenon*. Lehmer was able to prove the following: *if the Z -function has negative local maximum or positive local minimum, then the Riemann Hypothesis is false*; for proof see [Ivi13, Section 2.3]. In view of Lehmer's phenomenon this might happen for some large t . This, combined with some other evidence for complicated behavior of the Z -function, becomes a basic argument to oppose the validity of the Riemann Hypothesis. The reader can find more information in [BCRW08].

2010 *Mathematics Subject Classification.* 11M06, 11M26.

Key words and phrases. Hardy's Z -function, Riemann hypothesis, de Bruijn-Newman constant, Lehmer pair.

Different and perhaps more quantitative aspect of Lehmer's phenomenon related to the Riemann Hypothesis was discovered in 1994. It depends on Polya's approach with trigonometric integrals

$$H_t(z) := 2 \int_0^\infty \sum_{n=1}^\infty \left(2\pi^2 n^4 e^{9u/2} - 3\pi n^2 e^{5u/2} \right) e^{-\pi n^2 e^{2u} + tu^2} \cos(zu) du$$

for $t \in \mathbb{R}$ and $z \in \mathbb{C}$. It is standard that $H_0(z)$ has only real zeros if and only if the Riemann Hypothesis is true, see [Tit86, pp. 254-255]. An important theorem on this subject is: *There exists a real constant Λ , such that H_t has only real zeros if and only if $t \geq \Lambda$.* Then the Riemann Hypothesis is equivalent to the assertion $\Lambda \leq 0$. This constant is named after N. G. de Bruijn and C. M. Newman whose contributions [dB50] and [New76] culminated in the above theorem. Newman conjectured that $\Lambda \geq 0$; in his own words, "the Riemann Hypothesis, if true, is only barely so". Since then, an extensive research began to find smaller and smaller negative lower bounds of Λ supporting Newman's conjecture. The paper [CSV94] gives today's most promising method to find such bounds and it depends on a precise definition of Lehmer's phenomenon. It is reasonable to assume the Riemann Hypothesis, since its negation $\Lambda > 0$ implies Newman's conjecture. Then our definition, which uses the Z -function, becomes slightly simpler than the original one. Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the Z -function and define

$$\bar{g}_{\{\gamma_1, \gamma_2\}} := (\gamma_1 - \gamma_2)^2 \sum_{\gamma \notin \{\gamma_1, \gamma_2\}} \frac{1}{(\gamma - \gamma_1)^2} + \frac{1}{(\gamma - \gamma_2)^2} \quad (1)$$

where γ goes through all zeros of the Z -function.

Definition 1. A pair $\{\gamma_-, \gamma_+\}$ of two *consecutive* simple positive zeros of the Z -function is said to be a *Lehmer pair* if we have $\bar{g}_{\{\gamma_-, \gamma_+\}} < 4/5$.

For a Lehmer pair $\{\gamma_-, \gamma_+\}$ define

$$\lambda_{\{\gamma_-, \gamma_+\}} := \frac{(\gamma_+ - \gamma_-)^2}{2\bar{g}_{\{\gamma_-, \gamma_+\}}} \left(\left(1 - \frac{5}{4}\bar{g}_{\{\gamma_-, \gamma_+\}} \right)^{\frac{4}{5}} - 1 \right). \quad (2)$$

The main result of [CSV94] is $\lambda_{\{\gamma_-, \gamma_+\}} \leq \Lambda$. The Taylor series of (2) gives us

$$\left(-\frac{1}{2} - \frac{5}{32}\bar{g}_{\{\gamma_-, \gamma_+\}} \right) (\gamma_+ - \gamma_-)^2 < \lambda_{\{\gamma_-, \gamma_+\}} < -\frac{1}{2} (\gamma_+ - \gamma_-)^2. \quad (3)$$

Lehmer pairs are not so rare as someone might expect; the first thousand zeros contain 48 Lehmer pairs. In view of (3) closer zeros may produce better estimates in favour Newman's conjecture. But an infinite number of Lehmer pairs implies Newman's conjecture. This follows from the well-known theorem due to Littlewood (see Titchmarsh's beautiful proof [Tit86, pp. 224-226]), which asserts that the gaps between the ordinates of successive zeros of the ζ -function tend to zero. In [CSV94] it is proved that $\{\rho_{6709}, \rho_{6710}\}$ is really a Lehmer pair, and from this it follows $\Lambda > -7.113 \cdot 10^{-4}$. Their proof is done by estimating (1) with the help of nearby zeros and the knowledge of asymptotic behavior of a number of "far-away" zeros. This technique becomes essential in all further progress in obtaining better bounds; some of this is discussed in Section 3.

In order to produce "analytic" definition of Lehmer pairs, Stopple gives in [Sto16] more restrictive definition in terms of the first three derivatives of the Riemann xi-function $\Xi(t)$ at pair's zeros; see Section 2 for his result. He connects this to the location of the zeros of ζ' and uses similar methods as we mentioned above. Naïve question by the present author was if it is possible to prove that a given pair is Lehmer by simply computing derivatives. Unfortunately, values of the Ξ -function are very small, even for small t , therefore inappropriate for numerical

calculations. Using Stopple's method we prove the following estimation of (1) by means of derivatives of the Z -function.

Theorem 1. *Let $Z(t)$ be Hardy's Z -function and define the real function \widehat{F} by*

$$\widehat{F}(t) := -\frac{Z'''}{Z'}(t) + \frac{3}{4} \left(\frac{Z''}{Z'} \right)^2 (t). \quad (4)$$

Let $\{\gamma_1, \gamma_2\}$ be a pair of two simple zeros of the Z -function and define

$$\hat{g}_{\{\gamma_1, \gamma_2\}} := \frac{1}{3} (\gamma_1 - \gamma_2)^2 (\widehat{F}(\gamma_1) + \widehat{F}(\gamma_2)) - 2.$$

Under the Riemann Hypothesis we have

$$0 < \bar{g}_{\{\gamma_1, \gamma_2\}} - \hat{g}_{\{\gamma_1, \gamma_2\}} < 3 (\gamma_1 - \gamma_2)^2 \left(\frac{1}{\gamma_1^2} + \frac{1}{\gamma_2^2} \right). \quad (5)$$

This estimation is obviously very good for consecutive and relatively large zeros. The following immediate corollary gives conditions on a pair of zeros to be a Lehmer pair.

Corollary 1. *Notations and assumptions as in Theorem 1. If $\{\gamma_-, \gamma_+\}$ is a Lehmer pair, then*

$$(\gamma_+ - \gamma_-)^2 (\widehat{F}(\gamma_-) + \widehat{F}(\gamma_+)) < \frac{42}{5}.$$

If

$$(\gamma_+ - \gamma_-)^2 (\widehat{F}(\gamma_-) + \widehat{F}(\gamma_+)) < \frac{42}{5} - 3 (\gamma_+ - \gamma_-)^2 \left(\frac{1}{\gamma_-^2} + \frac{1}{\gamma_+^2} \right), \quad (6)$$

then $\{\gamma_-, \gamma_+\}$ is a Lehmer pair.

In Section 3 we demonstrate that it is possible to calculate the derivatives of the Z -function by simple rules of numerical integration (as composite trapezoidal rule is), quick enough even for quite large zeros. Therefore, Corollary 1 may serve as an alternative approach for deciding whether a pair of consecutive simple zeros is a Lehmer pair.

Before proceeding to the proof of Theorem 1, we observe that the expression on the right hand side of (6) is very close to $42/5$ for large zeros, especially when we have a good candidate for a Lehmer pair. Therefore, it is reasonable to conjecture that we can omit this small term without changing the conclusion of Corollary 1.

2. PROOF OF THEOREM 1

Recall one of equivalent definitions of Euler's gamma function

$$\Gamma(z) = \frac{e^{-\gamma z}}{z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n} \right)^{-1} e^{z/n} \quad (7)$$

where $\gamma \approx 0.5772$ is the Euler-Mascheroni constant. The gamma function is a meromorphic function with simple poles at $z = 0, -1, -2, \dots$ and without zeros.

Define the function H by

$$H(s) := \frac{1}{2} (1-s) s \pi^{-s/2} \Gamma\left(\frac{s}{2}\right). \quad (8)$$

This function is holomorphic in $\mathbb{C} \setminus \{-2n : n \in \mathbb{N}\}$ with simple zero at $s = 1$ and simple poles at negative even integers. Define the Riemann xi-function by $\xi(s) := H(s)\zeta(s)$. An important observation is that $\xi(s)$ is entire function with exactly those zeros which are nontrivial zeros of ζ -function. Also, simple zeros of the ξ -function correspond to simple zeros of the ζ -function. The xi-function along the critical line is also known as the Riemann xi-function, but writing with capital

xi, i. e., $\Xi(t) := \xi(1/2 + it)$. A nontrivial connection with trigonometric integrals mentioned in the introduction is $H_0(t) = \Xi(t)/2$, identity valid even for complex t .

It is well-known that the ξ -function can be written as the infinite product

$$\xi(s) = e^{Bs} \prod_{\rho} \left(1 - \frac{s}{\rho}\right) e^{s/\rho} \quad (9)$$

where ρ runs through all zeros of ξ -function and B is some constant. Such product was already conjectured by Riemann, but first proved by Hadamard in 1893. Let ρ_0 be some simple zero of $\xi(s)$. From (9) it follows that

$$\xi(s) = -(s - \rho_0) \prod_{\rho \neq \rho_0} e^{Bs+s/\rho_0} \rho_0^{-1} \left(1 - \frac{s}{\rho}\right) e^{s/\rho}. \quad (10)$$

The following lemma is crucial in Stopple's approach and also in ours. Its proof is very simple, but for convenience of the reader we include it here.

Lemma 1. *Assume that $f(s)$ and $g(s)$ are holomorphic functions on some domain $\Omega \subseteq \mathbb{C}$ such that $f(s) = (s - z)g(s)$ and $g(z) \neq 0$ for some $z \in \Omega$. Then*

$$\left(\left(\frac{f''}{f'} \right)' + \frac{1}{4} \left(\frac{f''}{f'} \right)^2 \right) (z) = 3 \left(\frac{g'}{g} \right)' (z).$$

Proof. We have $f'(s) = g(s) + (s - z)g'(s)$, $f''(s) = 2g'(s) + (s - z)g''(s)$ and $f'''(s) = 3g''(s) + (s - z)g'''(s)$. Therefore

$$\left(\frac{f''}{f'} \right)' (z) = \frac{f'''}{f'}(z) - \left(\frac{f''}{f'} \right)^2 (z) = 3 \frac{g''}{g}(z) - 4 \left(\frac{g'}{g} \right)^2 (z)$$

and we really get desired equality. ■

Applying Lemma 1 on product representation (10) of ξ -function, we get

$$\begin{aligned} F(\rho_0) &:= \frac{1}{3} \left(\left(\frac{\xi''}{\xi'} \right)' + \frac{1}{4} \left(\frac{\xi''}{\xi'} \right)^2 \right) (\rho_0) = - \sum_{\rho \neq \rho_0} (\rho - \rho_0)^{-2} \\ &= \frac{1}{3} \left(\frac{\xi'''}{\xi'} - \frac{3}{4} \left(\frac{\xi''}{\xi'} \right)^2 \right) (\rho_0). \end{aligned} \quad (11)$$

If we take two simple zeros ρ_1 and ρ_2 and define $\Delta := \rho_1 - \rho_2$, then (11) gives us

$$\Delta^2 \sum_{\rho \notin \{\rho_1, \rho_2\}} \frac{1}{(\rho - \rho_1)^2} + \frac{1}{(\rho - \rho_2)^2} = -\Delta^2 (F(\rho_1) + F(\rho_2)) - 2 \quad (12)$$

The left hand side of (12) is very similar to expression (1). Indeed, if we assume the Riemann Hypothesis, then $\bar{g}_{\{\gamma_1, \gamma_2\}} = (\gamma_1 - \gamma_2)^2 (F(\gamma_1) + F(\gamma_2)) - 2$ where

$$F(t) = -\frac{1}{3} \left(\left(\frac{\Xi''}{\Xi'} \right)' + \frac{1}{4} \left(\frac{\Xi''}{\Xi'} \right)^2 \right) (t).$$

Writing $P\Xi'(t) := (\Xi''/\Xi')'(t)$ for *pre-Schwarzian derivative* of Ξ' , the assertion $-(\gamma_- - \gamma_+)^2 (P\Xi'(\gamma_-) + P\Xi'(\gamma_+)) < 42/5$ implies that $\{\gamma_-, \gamma_+\}$ is a Lehmer pair. Stopple named such pair a *strong Lehmer pair* and showed by concrete calculations that this is indeed a much stronger condition.

We would like to express function F firstly in terms of the ζ -function and then in terms of the Z -function. Taking $\zeta(\rho_0) = 0$ into account, we simply differentiate

$\xi(s) = H(s)\zeta(s)$ to find that

$$\begin{aligned}\frac{\xi'''}{\xi'}(\rho_0) &= 3\frac{H''}{H}(\rho_0) + 3\frac{H'\zeta''}{H\zeta'}(\rho_0) + \frac{\zeta'''}{\zeta'}(\rho_0), \\ \frac{3}{4}\left(\frac{\xi''}{\xi'}\right)^2(\rho_0) &= 3\left(\frac{H'}{H}\right)^2(\rho_0) + 3\frac{H'\zeta''}{H\zeta'}(\rho_0) + \frac{3}{4}\left(\frac{\zeta''}{\zeta'}\right)^2(\rho_0)\end{aligned}$$

and finally

$$F(\rho_0) = \left(\frac{H'}{H}\right)'(\rho_0) + \frac{1}{3}\widetilde{F}(\rho_0) \quad \text{where} \quad \widetilde{F}(\rho_0) := \left(\frac{\zeta'''}{\zeta'} - \frac{3}{4}\left(\frac{\zeta''}{\zeta'}\right)^2\right)(\rho_0). \quad (13)$$

By (8) we obtain

$$\left(\frac{H'}{H}\right)'(\rho_0) = (\log H)''(\rho_0) = -\frac{1}{(1-\rho_0)^2} - \frac{1}{\rho_0^2} + \frac{1}{4}\zeta\left(2, \frac{\rho_0}{2}\right) \quad (14)$$

where $\zeta(s, z) := \sum_{n=0}^{\infty} (z+n)^{-s}$ is the *Hurwitz zeta-function*; it is a generalization of the Riemann zeta function since $\zeta(s, 0) = \zeta(s)$. To prove (14) observe that the second derivative of $\log \Gamma(z)$ is equal to $\zeta(2, z)$. Before proceeding to the proof of Theorem 1, we need to estimate the real part of the Hurwitz zeta-function in (14) for ρ_0 on the critical line.

Lemma 2. *For $t > 1$ we have*

$$\frac{-56t^2}{(4t^2+1)^2} < \Re \left\{ \zeta\left(2, \frac{1}{4} + i\frac{t}{2}\right) \right\} < \frac{3}{2t^2}. \quad (15)$$

Proof. Choose an arbitrary $t > 1$. It is easy to see that $\Re \{ \zeta(2, 1/4 + it/2) \} = \sum_{n=0}^{\infty} f(n)$ where

$$f(n) := 16 \left((1+4n)^2 - 4t^2 \right) \left((1+4n)^2 + 4t^2 \right)^{-2}.$$

We are interested in the behavior of this function on the interval $[0, \infty)$; at $n = 0$ it is negative, on the interval $[0, (2\sqrt{3}t - 1)/4]$ it increases to positive maximum $1/2t^{-2}$, then it decreases to zero at infinity. Because of such not so simple behavior we have

$$\sum_{n=0}^{\infty} f(n) > f(0) + \int_0^{\infty} f(n)dn - \frac{1}{2t^2} = -\frac{(4t^2-1)(28t^2-1)}{2t^2(4t^2+1)^2}$$

from which the left side of (15) follows. On the other hand,

$$\sum_{n=0}^{\infty} f(n) < \int_0^{\infty} f(n)dn + \frac{1}{2t^2} < \frac{3}{2t^2}$$

and the proof is complete. ■

Proof of Theorem 1. Straightforward calculations with $Z(\gamma_0) = 0$ in mind give us

$$\begin{aligned}\zeta'(\rho_0) &= -ie^{-i\vartheta(\gamma_0)}Z'(\gamma_0), \\ \zeta''(\rho_0) &= e^{-i\vartheta(\gamma_0)}(2i\vartheta'(\gamma_0)Z'(\gamma_0) - Z''(\gamma_0)), \\ \zeta'''(\rho_0) &= e^{-i\vartheta(\gamma_0)}((3\vartheta''(\gamma_0) - 3i\vartheta'^2(\gamma_0))Z'(\gamma_0) + 3\vartheta'(\gamma_0)Z''(\gamma_0) + iZ'''(\gamma_0)).\end{aligned}$$

From this, (13) and (4) we obtain $\widetilde{F}(\rho_0) = \widehat{F}(\gamma_0) + i3\vartheta''(\gamma_0)$ where this is indeed the decomposition into the real and the imaginary part of $\widetilde{F}(\rho_0)$. By (14) we have

$$\left(\frac{H'}{H}\right)'(\rho_0) = \frac{2\gamma_0^2 - 1/2}{(\gamma_0^2 + 1/4)^2} + \frac{1}{4}\zeta\left(2, \frac{1}{4} + i\frac{\gamma_0}{2}\right).$$

Since $\Im \{\zeta(2, \rho_0/2)\} = -4\vartheta''(\gamma_0)$ by the definition of the Hurwitz zeta-function and the function ϑ , $F(\rho_0) = (1/3)\widehat{F}(\gamma_0) + \varepsilon(\gamma_0)$ where

$$\varepsilon(\gamma_0) := \frac{2\gamma_0^2 - 1/2}{(\gamma_0^2 + 1/4)^2} + \frac{1}{4} \Re \left\{ \zeta \left(2, \frac{1}{4} + i \frac{\gamma_0}{2} \right) \right\}.$$

By Lemma 2 we have $0 < \varepsilon(\gamma_0) < 3\gamma_0^{-2}$, and by (12) we have

$$\bar{g}_{\{\gamma_1, \gamma_2\}} = \hat{g}_{\{\gamma_1 - \gamma_2\}} + (\gamma_1 - \gamma_2)^2 (\varepsilon(\gamma_1) + \varepsilon(\gamma_2)).$$

Combination of both gives inequality (5). ■

Using some results on gaps between consecutive zeros of the ζ -function, it is possible to obtain lower bound of (6). Let $0 < \gamma'_1 \leq \gamma'_2 \leq \gamma'_3 \leq \dots$ denote ordinates of zeros of the ζ -function in the upper half-plane; under the Riemann Hypothesis these ordinates are exactly zeros of the Z -function in $[0, \infty)$. In the appendix of this paper we prove

$$\gamma'_{n+1} - \gamma'_n \leq \gamma'_2 - \gamma'_1 \quad (16)$$

for every $n \in \mathbb{N}$. Since $\gamma'_2 - \gamma'_1 < \gamma'_1/2$ by concrete calculations, it follows $\gamma'_{n+1} - \gamma'_n < \gamma'_n/2$ for every $n \in \mathbb{N}$. Assuming the Riemann Hypothesis, this implies $\bar{g}_{\{\gamma_1, \gamma_2\}} > 4(\gamma_1 - \gamma_2)^2 (\gamma_1^{-2} + \gamma_2^{-2})$ for every distinct zeros $\gamma_1, \gamma_2 > \gamma'_1$ of the Z -function. Therefore, by (5) we have

$$6 < (\gamma_1 - \gamma_2)^2 (\widehat{F}(\gamma_1) + \widehat{F}(\gamma_2)) \quad (17)$$

for every two simple zeros γ_1 and γ_2 of the Z -function.

3. NUMERICAL RESULT

In 1986 van de Lune, te Riele and Winter found a spectacularly close pair of zeros, namely

$$\begin{aligned} \gamma_K &= 388\,858\,886.002\,285\,12, \\ \gamma_{K+1} &= 388\,858\,886.002\,393\,69 \end{aligned}$$

where $K = 1\,048\,449\,114$. By this pair of zeros and by the method of [CSV94] it was calculated in [COSV93] that $\Lambda > -5.895 \cdot 10^{-9}$. In [SGD11] the authors state that the probability of appearance of $\{\gamma_K, \gamma_{K+1}\}$ amongst the first $1.5 \cdot 10^9$ zeros is only 0.049; thus this pair is really exceptional. Also in [SGD11] the bound of Λ was improved to current record $-1.145 \cdot 10^{-11}$.

Let f be a holomorphic function on a domain $\Omega \subseteq \mathbb{C}$. Take arbitrary $z \in \Omega$ and $r > 0$ such that the closed disc $\{\zeta \in \mathbb{C}: |\zeta - z| \leq r\}$ is contained in domain Ω . By Cauchy's integral formula we have

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{|\zeta - z| = r} \frac{f(\zeta) d\zeta}{(\zeta - z)^{n+1}}. \quad (18)$$

From the definition of $Z(t)$ it follows that it is possible analytically continue the Z -function on right half-plane $\{z \in \mathbb{C}: \Re\{z\} > 0\}$. Applying (18) to the Z -function, and observing that derivatives of $Z(x)$ are real when x is real, gives us

$$Z^{(n)}(x) = \frac{n!}{r^n} \int_0^1 e^{-2\pi nit} Z(x + re^{2\pi it}) dt = \frac{n!}{r^n} \int_0^1 G(t) dt \quad (19)$$

where

$$G(t) := \Re \{ Z(x + re^{2\pi it}) \} \cos(2\pi nt) + \Im \{ Z(x + re^{2\pi it}) \} \sin(2\pi nt)$$

is a real function.

We used *Mathematica*'s function `RiemannSiegelZ` for the approximate calculation of the integral (19) by composite trapezoidal rule with tolerance 10^{-7} . Results we get for a pair $\{\gamma_K, \gamma_{K+1}\}$ are summarized in the following table.

	Z'	Z''	Z'''
γ_K	-0.008173	150.552	123.981
γ_{K+1}	0.008173	150.565	122.665

From this we obtain $(\gamma_K - \gamma_{K+1})^2 (\widehat{F}(\gamma_K) + \widehat{F}(\gamma_{K+1})) = 6.0001$, thus concluding that $\{\gamma_K, \gamma_{K+1}\}$ is really a Lehmer pair. Observe that this value is very close to the bound given by (17) and violation of this bound would disprove the Riemann Hypothesis. Maybe this is not so surprising because in view of Lehmer's theorem this pair of zeros is "nearly a counter-example" to the Riemann Hypothesis.

4. APPENDIX

Denote by $N(T)$ the number of γ'_n 's not exceeding T . One form of the Riemann-von Mangoldt formula is

$$\left| N(T) - \frac{T}{2\pi} \log \frac{T}{2\pi e} - \frac{7}{8} \right| \leq a \log T + b \log \log T + c + \frac{1}{5T} \quad (20)$$

for $T \geq e$ and suitable positive constants a, b, c . Current best result is $a = 0.110$, $b = 0.290$ and $c = 2.290$, see [PT15]. From (20) we obtain

$$\begin{aligned} N(T+H) - N(T) &> \left(\frac{H}{2\pi} - 2a \right) \log(T+H) - 2b \log \log(T+H) \\ &\quad + \frac{T}{2\pi} \log \left(1 + \frac{H}{T} \right) - \frac{H}{2\pi} \log(2\pi e) - 2c - \frac{2}{5T}. \end{aligned} \quad (21)$$

Therefore, for $H > 4\pi a \approx 1.38$ a constant T_0 exists such that $\gamma'_{n+1} - \gamma'_n \leq H$ for every $\gamma'_n > T_0$. Main advantage of (21) is that we can calculate T_0 for a given H .

Since $\gamma'_2 - \gamma'_1 < 7$, it is enough to demonstrate

$$\gamma'_{n+1} - \gamma'_n < H = 6 \quad \text{for } n \geq 2 \quad (22)$$

in order to prove (16). With help of *Mathematica* we obtain from (21) the bound $T_0 = 35370$; just for comparison, $H = 1.4$ gives $T_0 = 4.7 \cdot 10^{1497}$. Furthermore, it is known (see [Tru14, p. 281]) that for $T \in [0, 6.8 \cdot 10^6]$ inequality (20) is true for constants $a = b = 0$ and $c = 2$. This fact lower our bound to $T_0 = 412$. Up to this height we easily verified (22) by computer.

Acknowledgements. We would like to thank Jeffrey Stopple for his interest in the subject, and Aleksandar Ivić and Timothy Trudgian for pointing out references concerning gaps between consecutive zeros of the Riemann zeta function.

REFERENCES

- [BCRW08] P. Borwein, S. Choi, B. Rooney, and A. Weirathmueller, *The Riemann hypothesis: A resource for the aficionado and virtuoso alike*, CMS Books in Mathematics/Ouvrages de Mathématiques de la SMC, Springer, New York, 2008.
- [COSV93] G. Csordas, A. M. Odlyzko, W. Smith, and R. S. Varga, *A new Lehmer pair of zeros and a new lower bound for the de Bruijn-Newman constant Λ* , Electron. Trans. Numer. Anal. **1** (1993), 104–111.
- [CSV94] G. Csordas, W. Smith, and R. S. Varga, *Lehmer pairs of zeros, the de Bruijn-Newman constant Λ , and the Riemann hypothesis*, Constr. Approx. **10** (1994), no. 1, 107–129.
- [dB50] N. G. de Bruijn, *The roots of trigonometric integrals*, Duke Math. J. **17** (1950), 197–226.
- [Edw01] H. M. Edwards, *Riemann's zeta function*, Dover Publications, New York, 2001.
- [Ivi13] A. Ivić, *The theory of Hardy's Z-function*, Cambridge tracts in mathematics 196, Cambridge University Press, New York, 2013.

- [New76] C. M. Newman, *Fourier transforms with only real zeros*, Proc. Amer. Math. Soc. **61** (1976), no. 2, 245–251.
- [PT15] D. J. Platt and T. S. Trudgian, *An improved explicit bound on $|\zeta(1/2 + it)|$* , J. Number Theory **147** (2015), 842–851.
- [SGD11] Y. Saouter, X. Gourdon, and P. Demichel, *An improved lower bound for the de Bruijn–Newman constant*, Math. Comp. **80** (2011), no. 276, 2281–2287.
- [Sto16] J. Stopple, *Lehmer pairs revisited*, Experimental Mathematics (2016), DOI: 10.1080/10586458.2015.1107870.
- [Tit86] E. C. Titchmarsh, *The theory of the Riemann zeta-function*, 2nd ed., Oxford University Press, New York, 1986.
- [Tru14] T. S. Trudgian, *An improved upper bound for the argument of the Riemann zeta-function on the critical line II*, J. Number Theory **134** (2014), 280–292.

FACULTY OF MATHEMATICS AND PHYSICS, UNIVERSITY OF LJUBLJANA, JADRANSKA 19, 1000 LJUBLJANA, SLOVENIA

E-mail address: `aleksander.simonc@student.fmf.uni-lj.si`